Hedonic Games with Social Context

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Abstract. Hedonic games are coalition formation games in which coalitions are created as a result of the strategic interaction of independent players. To this day, the literature on non-cooperative hedonic games has considered totally selfish players; our aim is that of defining and studying a new model in which, given a social graph, players also care about the happiness of their friends: we call this class of games *social context hedonic games (SCHGs)*. We consider Nash equilibria of SCHGs, and study their existence, convergence and performance with respect to the classical notions of price of anarchy and price of stability. In particular, we provide an exact potential function for SCHGs implying the existence and convergence to Nash equilibria, and we prove tight or asymptotically tight bounds on the price of anarchy and the price of stability of SCHGs.

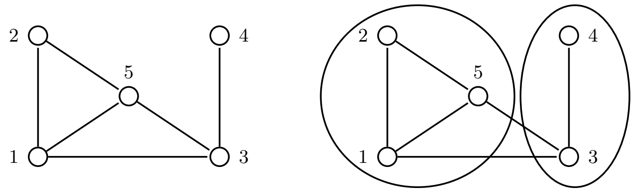
Keywords: Coalition Formation; Hedonic Games; Nash Equilibrium; Price of Anarchy; Price of Stability; Social Context

1 Introduction

An important issue in computer science is that of investigating the dynamics that regulates clustering and coalition formation. Hedonic games, introduced by Dréze and Greenberg [15], represent a framework for studying the formal aspects of the formation of player coalitions. In these games, players have preferences over the set of all possible player coalitions, and the utility of a player depends on the composition of the coalition she belongs to.

Hedonic games are of great interest because they model natural behavioral dynamics in social environments: in economic, social and political situation, in fact, individuals carry out activities in groups rather than by themselves. Politicians, for example, may want to be in a party that maximizes like-minded members or, more in general, people may want to be with people of the same ethnic or social group.

While the standard model of hedonic games assumes that players are totally selfish, in this paper we are interested in analyzing the case in which players take into account also the happiness of their friends. We call these games *Social Context Hedonic Games* (SCHGs); we believe that they provide a more realistic model for hedonic games, because they capture the fact that the behavior of players also depends on the happiness of their friends. To this aim, we consider



(i,j)	$v_{i,j}$
(1,3)	1
(2,5)	3
(1,5)	2
(3,4)	1

Fig. 1. A social network G, a coalition structure $\mathcal{C}$ and the non-null valuations $v_{i,j}$.

an underlying social network represented by a graph, whose nodes are players and in which an edge connecting two players expresses friendship between them. Arguably, as a first step in the study of SCHGs, it is more natural to consider symmetric friendship relations and therefore we focus on undirected graphs.

In SCHGs, valuations are additive and each player has a valuation v for any other player, but differently from the classical hedonic games, the utility of a coalition to a particular player is given not only by the sum of the valuations she assigns to the members of her coalition, but also depends on the sum of the valuations her friends assign to the members of their own coalitions (the latter contribution is multiplied by a given parameter $\alpha \in [0,1]$). In particular, for $\alpha = 0$ we model classical hedonic games in the totally selfish setting and when $\alpha = 1$ we can model a fully altruistic setting. In Figure 1, for example, we can see that the utility of player 5 is equal to $v_{1,5} + v_{2,5} + \alpha(v_{1,5} + v_{2,5} + v_{3,4}) = 2 + 3 + \alpha(2 + 3 + 1)$, where $2 + 3$ is the sum of valuations of the players in her coalition and $\alpha(2 + 3 + 1)$ is the sum of valuations her friends 1, 2 and 3 assign to the members of their own coalitions, multiplied by α.

Our aim is to study the existence and performance of natural stable outcomes for SCHGs. We will focus on *Nash stable* outcomes, i.e., outcomes in which no player can improve her utility by unilaterally changing her own coalition. In particular, we evaluate the performance of Nash outcomes for SCHGs by means of the widely used notions of price of anarchy and price of stability, which are defined as the ratio between the social optimal value and the social value of the worst (resp. best) stable outcome.

1.1 Our Results

By providing an exact potential function for the SCHGs, we show that these games always posses a pure Nash equilibrium and also that the convergence to such stable outcomes is guaranteed. We consider two social welfare functions. The first social function, $\overline{\mathsf{SW}}$, is given by the summation, for each player, of the values she assigns to the members of her coalition, while the second social function, denoted by SW, is the summation of the players' utilities (taking into account, for any player, also the contribution due to the valuations of her friends multiplied by α). We evaluate, for both of them, the performance of the Nash outcomes by means of the notions of price of anarchy and price of stability (PoA and $\overline{\mathsf{PoA}}$ denote the price of anarchy with respect to SW and $\overline{\mathsf{SW}}$, respectively;

analogously PoS and $\overline{\mathsf{PoS}}$ denote the price of stability with respect to SW and $\overline{\mathsf{SW}}$, respectively).

In presence of negative valuations, both PoS and $\overline{\mathsf{PoS}}$ (and therefore also PoA and $\overline{\mathsf{PoA}}$) can be unbounded. Furthermore, in some cases we are able to provide instances in which the social value of any equilibrium $\mathcal{C}$ is negative while the optimal solution lead to a positive outcome.

We subsequently turn our attention to the case of non-negative valuations and we prove that the price of anarchy is $\Theta(n)$, while the price of stability is 1.

1.2 Related Work

Social context games are introduced in [2]. These games are defined by an underlying game in strategic form, and a social context consisting of an undirected graph and an aggregation function. The authors consider resource selection games as the underlying game and they study the existence of pure strategy Nash equilibrium. Building on this model, Bilò et al. [8] investigate social context games in which the underlying games are linear congestion games and Shapley cost sharing games, while the aggregation functions are min, max and sum. Moreover, Anagnostopoulos et al. [1] study the effects of the altruistic behaviour of players showing that the price of anarchy may increase as the players become more altruistic. They show that this increase is modest for congestion games and min-sum scheduling games, whereas it might be drastic for generalized second price auctions. The interests on altruistic players have been also modelled and studied by Hoefer and Skopalik [19]: they focus on the existence and complexity of pure Nash equilibria with altruistic agents in atomic congestion games. Chen et al. [13] study the inefficiency of equilibria for several classes of games such as cost-sharing games, utility games, and linear congestion games. Salehi-Abari and Boutilier [25] study social choice with empathetic preferences and their local empathetic model is related to the model presented in [21]. Finally, Brânzei and Larson [11] study social distance games. In these games a players opinion on her friends (players of distance one) has the highest weight while her opinion on players farther away counts less.

Several papers are devoted to the study of hedonic games. They are introduced by Dréze and Greenberg [15], who analyze hedonic games under a cooperative perspective. Properties guaranteeing the existence of core allocations for games with additively separable utility have been studied by Banerjee, Konishi and Sönmez [7], while Bogomolnaia and Jackson [10] deal with several forms of stable outcomes like the core, Nash and individual stability. Ballester [4] considers computational complexity issues related to hedonic games, and shows that the core and the Nash stable outcomes have corresponding NP-complete decision problems for a variety of situations, while Aziz et al. [3] study the computational complexity of stable coalitions in additively separable hedonic games. Moreover, Olsen [22] proves that the problem of deciding whether a Nash stable coalitions exists in an additively separable hedonic game is NP-complete, as well as the one of deciding whether a non-trivial Nash stable coalitions exists in an additively separable hedonic game with non-negative and symmetric preferences (i.e., unweighted undirected graphs). Feldman et al. [16] investigate some interesting subclasses of hedonic games from a non-cooperative point of view, by characterizing Nash equilibria and providing upper and lower bounds on both the price of stability and the price of anarchy. In their model the agents lie in a metric space with a distance function modeling their distance or "similarity". Peters [23] considers "graphical" hedonic games where agents form the vertices of an undirected graph, and each agent's utility function only depends on the actions taken by her neighbors (with general value functions). Moreover, hedonic games have also been considered by Charikar et al. [12] and by Demaine et al. [14] from a classical optimization point of view (i.e, without requiring stability for the solutions) and by Flammini et al. in an online setting [17]. Peters et al. [24] consider several classes of hedonic games and identify simple conditions on expressivity that are sufficient for the problem of checking whether a given game admits a stable outcome to be computationally hard. From a different perspective, strategyproof mechanisms for additively separable hedonic and fractional hedonic games have been proposed in [18], while stable outcomes for these games and for modified fractional hedonic games are presented in [9] and [20]. Finally, hedonic games are being widely investigated also under different utility definitions. For instance, in [5, 6], coalition formation games in which agent utilities are proportional to their harmonic centralities in the respective coalitions are considered.

To the best of our knowledge, few papers deal with the notion of altruism in hedonic games. Nguyen et al. [21] define altruistic hedonic games where the satisfaction of players' friends is taken into account according to three degrees of altruism, from being selfish first, over aggregating opinions of a player and her friends equally, to altruistically letting ones friends decide first. They study both the axiomatic properties of these games and the computational complexity of problems related to common stability concepts. In [26], Umar and Mesbah model the problem of joint coalition formation and bandwidth allocation in ad hoc radio networks made of selfish/altruistic nodes as a hedonic coalition formation game with non-transferable utility. The authors study the computational complexity and convergence properties of the proposed hedonic algorithm under selfish and altruistic preferences, and present means to guarantee Nash-stability.

The paper is organized as follows. In Section 2 we formally define the hedonic games with social context. The technical contributions of the paper are then presented in Sections 3, 4 and 5 which address the existence of Nash outcomes, the results on the price of anarchy and those on the price of stability, respectively. Finally, in Section 6 we list some interesting open problems. Due to space limitations, some proofs are omitted.

2 Model

For an integer $k > 0$, denote with $[k]$ the set $\{1, \dots, k\}$.

We model a Social Context Hedonic Game by means of a valuation function v, an undirected graph $G = (N, E)$ and a given parameter $\alpha \in [0, 1]$. We denote with $n = |N|$ the number of nodes of G and with E the set of edges between the nodes, that represent the friendship relation. $v : N \times N \to \mathcal{R}$ is the symmetric valuation function. For the sake of convenience, we adopt the notation (i, j) and $v_{i,j}$ to denote the pair $\{i, j\} \in N \times N$ and its valuation $v(\{i, j\})$, respectively.

Given a symmetric valuation function v, an undirected graph $G = (N, E)$ and a value for α, the *Social Context Hedonic Game* induced by G, v and α, denoted as $\mathcal{G}(G, v, \alpha)$, is the game in which each node $i \in N$ is associated with a player. We assume that players are numbered from 1 to n and, for every $i \in [n]$, each player chooses to join a certain *coalition* among n candidate ones: the strategy of player i is an integer $j \in [n]$, meaning that player i is selecting candidate coalition C_j. A coalition structure (also called outcome) is a partition of the set of players into n coalitions $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ such that $C_j \subseteq N$ for each $j \in [n]$, $\bigcup_{j\in[n]} C_j = N$ and $C_i \cap C_j = \emptyset$ for any $i, j \in [n]$ with $i \neq j$. Notice that, since the number of candidate coalitions is equal to the number of players (nodes), some coalition may be empty. If $i \in C_j$, we say that player i is a member of the coalition C_j. We denote by $\mathcal{C}(i)$ the coalition in $\mathcal{C}$ of which player i is a member. In an outcome $\mathcal{C}$, the utility of player i is defined as

$$u_i(\mathcal{C}) = \overline{u_i}(\mathcal{C}) + \alpha \cdot \sum_{(i,j)\in E} \overline{u_j}(\mathcal{C}),$$

where, for every $i \in [n]$, $\overline{u_i}(\mathcal{C}) = \sum_{j\in\mathcal{C}(i)} v_{i,j}$.

Each player chooses the coalition she belongs to with the aim of maximizing her utility. We denote by $(\mathcal{C}, i, j)$, the new coalition structure obtained from $\mathcal{C}$ by moving player i from $\mathcal{C}(i)$ to C_j; formally, $(\mathcal{C}, i, j) = \mathcal{C} \setminus \{\mathcal{C}(i), C_j\} \cup \{\mathcal{C}(i) \setminus \{i\}, C_j \cup \{i\}\}$. A player *deviates* if she changes the coalition she belongs to. Given an outcome $\mathcal{C}$, an *improving move* (or simply a *move*) for player i is a deviation to any coalition C_j that strictly increases her utility, i.e., $u_i((\mathcal{C}, i, j)) > u_i(\mathcal{C})$. Moreover, player i performs a *best-response* in coalition structure $\mathcal{C}$ by choosing a coalition providing her the highest possible utility (notice that a best-response is also a move when there exists a coalition C_j such that $u_i((\mathcal{C}, i, j)) > u_i(\mathcal{C})$. A player is *stable* if she cannot perform a move. An outcome is *(pure) Nash stable* (or a *Nash equilibrium*) if every player is stable. An *improving dynamics*, or simply a dynamics, is a sequence of moves, while a *best-response dynamics* is a sequence of best-responses. A game has the *finite improvement path property* if it does not admit an improvement dynamics of infinite length. Clearly, a game possessing the finite improvement path property always admits a Nash stable outcome. We denote with $\mathsf{N}(\mathcal{G}(G, v, \alpha))$ the set of Nash stable outcomes of $\mathcal{G}(G, v, \alpha)$.

The *social welfare* of a coalition structure $\mathcal{C}$ is the summation of the players' utilities, i.e., $\mathsf{SW}(\mathcal{C}) = \sum_{i\in N} u_i(\mathcal{C})$.

We define also a second social welfare function $\overline{\mathsf{SW}}(\mathcal{C}) = \sum_{i\in N} \overline{u_i}(\mathcal{C})$ which is given by the summation, for each player, of the valuations she assigns to the members of her coalition (without considering her friends' utilities).

Given a game $\mathcal{G}(G, v, \alpha)$, an *optimum* coalition structure $\mathcal{C}^*(\mathcal{G}(G, v, \alpha))$ (respectively $\overline{\mathcal{C}^*}(\mathcal{G}(G, v, \alpha)))$ is one that maximizes the social welfare SW (respectively $\overline{\mathsf{SW}}$) of $\mathcal{G}(G,v,\alpha)$. The *price of anarchy* of a social context hedonic game $\mathcal{G}(G,v,\alpha)$ is defined as the worst-case ratio between the social welfare of a social optimum outcome and that of a Nash equilibrium. Formally, for any $k = 1,\dots,n$, $\mathsf{PoA}(\mathcal{G}(G,v,\alpha)) = \max_{\mathcal{C}\in\mathsf{N}(\mathcal{G}(G,v,\alpha))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G,v,\alpha)))}{\mathsf{SW}(\mathcal{C})}$ and $\overline{\mathsf{PoA}}(\mathcal{G}(G,v,\alpha)) = \max_{\mathcal{C}\in\mathsf{N}(\mathcal{G}(G,v,\alpha))} \frac{\overline{\mathsf{SW}}(\overline{\mathcal{C}^*}(\mathcal{G}(G,v,\alpha)))}{\overline{\mathsf{SW}}(\mathcal{C})}$. Analogously, the *price of stability* of $\mathcal{G}(G,v,\alpha)$ is defined as the best-case ratio between the social welfare of a social optimum outcome and that of a Nash equilibrium. Formally, for any $k = 1,\dots,n$, $\mathsf{PoS}(\mathcal{G}(G,v,\alpha)) = \min_{\mathcal{C}\in\mathsf{N}(\mathcal{G}(G,v,\alpha))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G,v,\alpha)))}{\mathsf{SW}(\mathcal{C})}$ and $\overline{\mathsf{PoS}}(\mathcal{G}(G,v,\alpha)) = \min_{\mathcal{C}\in\mathsf{N}(\mathcal{G}(G,v,\alpha))} \frac{\overline{\mathsf{SW}}(\overline{\mathcal{C}^*}(\mathcal{G}(G,v,\alpha)))}{\overline{\mathsf{SW}}(\mathcal{C})}$.

3 Nash Stable Outcomes

In this section we consider Nash stable outcomes. We show that a stable outcome is guaranteed to exist and also that the finite improvement path property holds for SCHGs, because these games admit the potential function

$$\Phi(\mathcal{C}) = \frac{1}{2}\sum_{i\in N} \hat{u}_i(\mathcal{C}),$$

with $\hat{u}_i(\mathcal{C}) = \sum_{j\in\mathcal{C}(i)} v'_{i,j}$, where, for each pair of players (i,j), we define $v'_{i,j} = v_{i,j}\cdot(1+\alpha)$ if $(i,j)\in E$ and $v'_{i,j} = v_{i,j}$ if $(i,j)\notin E$.

Thus, we can rewrite the potential function $\Phi(\mathcal{C})$ as

$$\Phi(\mathcal{C}) = \sum_{j\in\mathcal{C}(i)} v_{i,j} + \alpha\cdot \sum_{j\in\mathcal{C}(i),(i,j)\in E} v_{i,j}.$$

In the following theorem, we prove that $\Phi(\mathcal{C})$ is an exact potential function for our game.

Theorem 1. *Φ is a potential function for SCHGs.*

Proof. Given two stable outcomes $\mathcal{C}$ and $\mathcal{C}'$ where $\mathcal{C}'$ is obtained form $\mathcal{C}$ after a player i performs a move, we prove that the following holds:

$$\Phi(\mathcal{C}') - \Phi(\mathcal{C}) = u_i(\mathcal{C}') - u_i(\mathcal{C}). \tag{1}$$

For the left hand side of Equation (1), by applying the definition of Φ, we obtain that:

$$\begin{aligned}
\Phi(\mathcal{C}') - \Phi(\mathcal{C}) &= \frac{1}{2}\left(\sum_{i\in N}\hat{u}_i(\mathcal{C}') - \sum_{i\in N}\hat{u}_i(\mathcal{C})\right) = \frac{1}{2}\sum_{i\in N}\left(\hat{u}_i(\mathcal{C}') - \hat{u}_i(\mathcal{C})\right)\\
&= \frac{1}{2}\cdot 2\left(\sum_{j\in\mathcal{C}'(i)} v'_{i,j} - \sum_{j\in\mathcal{C}(i)} v'_{i,j}\right)\\
&= \sum_{j\in\mathcal{C}'(i)} v_{i,j} + \alpha \sum_{j\in\mathcal{C}'(i),(i,j)\in E} v_{i,j} - \sum_{j\in\mathcal{C}(i)} v_{i,j} - \alpha \sum_{j\in\mathcal{C}(i),(i,j)\in E} v_{i,j}.
\end{aligned}$$

In the right hand side we obtain that:

$$
\begin{aligned}
u_i(\mathcal{C}') - u_i(\mathcal{C}) &= \bar{u}_i(\mathcal{C}') + \alpha \sum_{(i,j)\in E} \bar{u}_j(\mathcal{C}') - \bar{u}_i(\mathcal{C}) - \alpha \sum_{(i,j)\in E} \bar{u}_j(\mathcal{C}) \\
&= \bar{u}_i(\mathcal{C}') - \bar{u}_i(\mathcal{C}) + \alpha \sum_{(i,j)\in E} (\bar{u}_j(\mathcal{C}') - \bar{u}_j(\mathcal{C})) \\
&= \sum_{j\in\mathcal{C}'(i)} v_{i,j} + \alpha \sum_{j\in\mathcal{C}'(i),(i,j)\in E} v_{i,j} - \sum_{j\in\mathcal{C}(i)} v_{i,j} - \alpha \sum_{j\in\mathcal{C}(i),(i,j)\in E} v_{i,j}.
\end{aligned}
$$

Hence, the proof follows. □

4 Price of Anarchy

In this section we evaluate the performance of Nash stable outcomes with respect to the notion of price of anarchy.

We first show that the price of anarchy is unbounded for general valuations. The following result holds for both the social welfare functions SW and $\overline{\mathsf{SW}}$, even when the social graph has no edges.

Theorem 2. *For any $\alpha \in [0, 1]$, there exists a function v (also admitting negative valuations), such that $\mathsf{PoA}(\mathcal{G}(G, v, \alpha))$ and $\overline{\mathsf{PoA}}(\mathcal{G}(G, v, \alpha))$ are unbounded, where $G = (N, \emptyset)$.*

For more involved social graphs, we are also able to show a stronger result, holding even for the case of the price of stability (analyzed in Section 5): by Theorem 6, there exists a SCHG in which every Nash equilibrium $\mathcal{C}$ is such that $\mathsf{SW}(\mathcal{C})$ is negative, while $\mathsf{SW}(\mathcal{C}^*)$ is positive. We now prove a similar result for the social welfare function $\overline{\mathsf{SW}}$.

Theorem 3. *For any $\alpha \in (0, 1]$, there exists a graph G and a function v (also admitting negative valuations) inducing $\mathcal{G}(G, v, \alpha)$, such that $\overline{\mathsf{SW}}(\mathcal{C}^*) > 0$ while $\overline{\mathsf{SW}}(\mathcal{C}) < 0$ for a Nash stable outcome $\mathcal{C}$ of $\mathcal{G}(G, v, \alpha)$.*

Given these negative results, in what follows we focus on the case in which the valuation function does not assume negative values, i.e., $v_{i,j} \geq 0$ for any $i, j \in [n]$ with $i \neq j$.

In order to prove the upper bounds to PoA and $\overline{\mathsf{PoA}}$, we need some additional notation and definitions. Given any outcome $\mathcal{C}$, let $\delta_i(\mathcal{C})$ be the sum of the valuations of player i toward her friends belonging to $\mathcal{C}(i)$, i.e $\delta_i(\mathcal{C}) = \sum_{j\in\mathcal{C}(i):(i,j)\in E} v_{i,j}$, and, analogously, let $\bar{\delta}_i(\mathcal{C}) = \sum_{j\in\mathcal{C}(i):(i,j)\notin E} v_{i,j}$ be the sum of the valuations of player i toward players belonging to $\mathcal{C}(i)$ and not being her friends. Finally, we denote by δ_i^{max} the maximum valuation of player i, i.e. $\delta_i^{max} = \max_{j\in N} v_{i,j}$.

The following theorems provide asymptotically matching upper and lower bounds to PoA and $\overline{\mathsf{PoA}}$.

Theorem 4. *For any $\alpha \in [0,1]$, any graph G and any function v not admitting negative valuations, $\overline{\mathsf{PoA}}(\mathcal{G}(G,v,\alpha)) \leq (n-1)(1+\alpha)$ and $\mathsf{PoA}(\mathcal{G}(G,v,\alpha)) \leq (n-1)(1+\alpha)$.*

Proof. Given a Nash stable outcome $\mathcal{C}$, for every player $i \in [n]$ it holds that

$$\delta_i(\mathcal{C}) + \bar{\delta}_i(\mathcal{C}) + \alpha\delta_i(\mathcal{C}) \geq \delta_i^{max}.$$

In fact, recall that $u_i(\mathcal{C}) = \overline{u_i}(\mathcal{C}) + \alpha \cdot \sum_{(i,j)\in E} \overline{u_j}(\mathcal{C})$, where, for every $i \in [n]$, $\overline{u_i}(\mathcal{C}) = \sum_{j\in\mathcal{C}(i)} v_{i,j}$. By the definitions of $\delta_i(\mathcal{C})$ and $\bar{\delta}_i(\mathcal{C})$, $\overline{u_i}(\mathcal{C}) = \delta_i(\mathcal{C}) + \bar{\delta}_i(\mathcal{C})$. Moreover, let $\beta_i(\mathcal{C}) = \sum_{(i,j)\in E} \overline{u_j}(\mathcal{C}) - \delta_i(\mathcal{C})$: it follows that $u_i(\mathcal{C}) = \delta_i(\mathcal{C}) + \bar{\delta}_i(\mathcal{C}) + \alpha \cdot (\beta_i(\mathcal{C}) + \delta_i(\mathcal{C}))$. Notice that if player i changes her strategy by joining the coalition containing player j such that $v_{i,j} = \delta_i^{max}$, inducing in this way a new coalition structure $\mathcal{C}'$, we obtain $u_i(\mathcal{C}') \geq \delta_i^{max} + \alpha\beta_i(\mathcal{C})$, because the contributions $\beta_i(\mathcal{C})$ of the friends of i not connected to player i in $\mathcal{C}(i)$ remain unchanged in $\mathcal{C}'$. Therefore, if $\delta_i^{max} + \alpha\beta_i(\mathcal{C}) > \delta_i(\mathcal{C}) + \bar{\delta}_i(\mathcal{C}) + \alpha \cdot (\beta_i(\mathcal{C}) + \delta_i(\mathcal{C}))$ player i would increase her utility by changing her strategy: a contradiction to the fact that $\mathcal{C}$ is Nash stable.

Moreover, it trivially holds that in all coalition structures, including the optimal outcomes $\mathcal{C}^*$ and $\overline{\mathcal{C}^*}$, $\overline{u_i}$ is at most $(n-1)\cdot\delta_i^{max}$. Thus, $\overline{u_i}(\mathcal{C}^*) \leq (n-1)\cdot\delta_i^{max}$ and $\overline{u_i}(\overline{\mathcal{C}^*}) \leq (n-1)\cdot\delta_i^{max}$.

Therefore,

$$\frac{\overline{\mathsf{SW}}(\overline{\mathcal{C}^*})}{\overline{\mathsf{SW}}(\mathcal{C})} \leq \frac{(n-1)\cdot\sum_{i\in N}\delta_i^{max}}{\sum_{i\in N}\overline{u_i}(\mathcal{C})}.$$

Since $\overline{u_i}(\mathcal{C}) = \delta_i(\mathcal{C}) + \bar{\delta}_i(\mathcal{C}) \geq \frac{\delta_i^{max}}{(1+\alpha)}$, we obtain that

$$\frac{\overline{\mathsf{SW}}(\overline{\mathcal{C}^*})}{\overline{\mathsf{SW}}(\mathcal{C})} \leq \frac{(n-1)\cdot\sum_{i\in N}\delta_i^{max}}{\frac{\sum_{i\in N}\delta_i^{max}}{1+\alpha}} = (n-1)(1+\alpha).$$

Analogously, for the social welfare function SW, since $u_i(\overline{\mathcal{C}^*}) \leq (n-1)\cdot\delta_i^{max}$ and $u_i(\overline{\mathcal{C}^*}) \geq \frac{\delta_i^{max}}{(1+\alpha)}$, by recalling the definition of u_i, we obtain that

$$\begin{aligned}\frac{\mathsf{SW}(\mathcal{C}^*)}{\mathsf{SW}(\mathcal{C})} &\leq \frac{\sum_{i\in N}\left[(n-1)\delta_i^{max} + \alpha\sum_{j\in N:(i,j)\in E}(n-1)\delta_j^{max}\right]}{\sum_{i\in N}\left[\delta_i^{max} + \alpha\sum_{j\in N:(i,j)\in E}\delta_j^{max}\right]\frac{1}{1+\alpha}} \\ &= \frac{(n-1)\sum_{i\in N}\left[\delta_i^{max} + \alpha\sum_{j\in N:(i,j)\in E}\delta_j^{max}\right]}{\sum_{i\in N}\left[\delta_i^{max} + \alpha\sum_{j\in N:(i,j)\in E}\delta_j^{max}\right]\frac{1}{1+\alpha}} \\ &\leq (n-1)(1+\alpha),\end{aligned}$$

thus proving the claim. □

We now focus on the lower bound to PoA and $\overline{\mathsf{PoA}}$.

Theorem 5. *For every even positive integer n and every $\alpha \in [0,1]$, there exist a graph G with n vertices and a valuation function v such that $\overline{\mathsf{PoA}}(\mathcal{G}(G,v,\alpha)) \geq (1+\alpha)\frac{n}{2} - \alpha$ and $\mathsf{PoA}(\mathcal{G}(G,v,\alpha)) \geq (1+\alpha)\frac{n}{2} - \alpha$.*

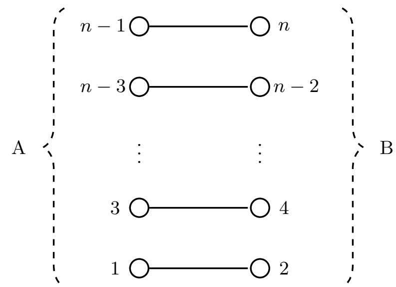


Fig. 2. The bipartite graph G.

Proof. Consider the bipartite graph $G = (A \cup B, E)$ with n vertices depicted in Figure 2 (note that $A = \{1, 3, \ldots, n-1\}$ and $B = \{2, 4, \ldots, n\}$), and let v the valuation function in which

- $v_{i,j} = v_{j,i} = \frac{1}{1+\alpha}$ for every $(i, j) \in E$;
- $v_{i,j} = v_{j,i} = 1$ for all pairs $(i, j) \notin E$ such that $i \in A$ and $j \in B$;
- $v_{i,j} = 0$ for all remaining pairs.

On the one hand, as it can be easily checked, the grand coalition $\mathcal{C}^o$ is such that, for every $i \in [n]$, $\overline{u_i}(\mathcal{C}^o) = \frac{n}{2} - 1 + \frac{1}{1+\alpha}$. Therefore, the optimal outcome $\overline{\mathcal{C}^*}$ is such that $\overline{\mathsf{SW}}(\overline{\mathcal{C}^*}) \geq n\left(\frac{n}{2} - 1 + \frac{1}{1+\alpha}\right)$ and the optimal outcome $\mathcal{C}^*$ is such that $\mathsf{SW}(\mathcal{C}^*) \geq n(1+\alpha)\left(\frac{n}{2} - 1 + \frac{1}{1+\alpha}\right)$.

On the other hand, the coalition structure $\mathcal{C}$ in which there are $\frac{n}{2}$ non-empty coalitions $\{i, i+1\}$ for $i = 1, 3, \ldots, n-1$ is a Nash stable outcome; in fact, for every $i \in A$, $\overline{u_i}(\mathcal{C}) = \frac{1}{1+\alpha}$ and $u_i(\mathcal{C}) = \frac{1}{1+\alpha} + \alpha \cdot \frac{1}{1+\alpha} = 1$, while a deviation of player i to another non-empty candidate coalition would induce a new coalition structure $\mathcal{C}'$ such that $u_i(\mathcal{C}') = 1 + \alpha \cdot 0 = 1$, because for player $i+1$ connected in G to player i it holds $\overline{u_{i+1}}(\mathcal{C}') = 0$. Moreover, a deviation of player i to an empty candidate coalition would induce a new coalition structure $\mathcal{C}''$ such that $u_i(\mathcal{C}'') = 0$, again because for player $i+1$ connected in G to player i it holds $\overline{u_{i+1}}(\mathcal{C}'') = 0$. A completely symmetric argument holds for every player $i \in B$.

It follows that

$$\frac{\overline{\mathsf{SW}}(\overline{\mathcal{C}^*})}{\overline{\mathsf{SW}}(\mathcal{C})} \geq \frac{n\left(\frac{n}{2} - 1 + \frac{1}{1+\alpha}\right)}{n\left(\frac{1}{1+\alpha}\right)} = (1+\alpha)\frac{n}{2} - \alpha.$$

Analogously, for the social welfare function SW, it holds that

$$\frac{\mathsf{SW}(\mathcal{C}^*)}{\mathsf{SW}(\mathcal{C})} \geq \frac{n\left(\frac{n}{2} - 1 + \frac{1}{1+\alpha}\right)(1+\alpha)}{n\left(\frac{1}{1+\alpha} + \frac{\alpha}{1+\alpha}\right)} = (1+\alpha)\frac{n}{2} - \alpha.$$

□

5 Price of Stability

In this section we present our results on the price of stability. First of all, notice that, if all valuations are non-negative, with respect to both social welfare functions SW and $\overline{\mathsf{SW}}$, the grand coalition is at the same time an optimal solution and a Nash stable outcome, thus implying that $\mathsf{PoS} = \overline{\mathsf{PoS}} = 1$.

Therefore, in the following we deal with the case of general valuations, i.e., we allow the valuation function to assume negative values. We first show that, for the social welfare function SW, there exists a SCHG in which every Nash equilibrium $\mathcal{C}$ is such that $\mathsf{SW}(\mathcal{C})$ is negative, while $\mathsf{SW}(\mathcal{C}^*)$ is positive.

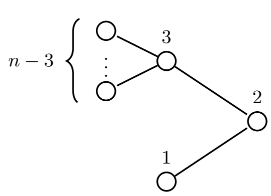
Theorem 6. *For any $\alpha \in (0,1]$, there exists a graph G and a function v (admitting also negative valuations) inducing $\mathcal{G}(G,v,\alpha)$, such that $\mathsf{SW}(\mathcal{C}^*) > 0$ while $\mathsf{SW}(\mathcal{C}) < 0$ for every Nash stable outcome $\mathcal{C}$ of $\mathcal{G}(G,v,\alpha)$.*

Proof. Let us consider the graph G depicted in Figure 3 and the valuation function v whose non-null values are listed in Figure 3, and in which $\epsilon < \alpha$ is an arbitrary positive parameter.

Now we want to show that in every Nash stable outcome $\mathcal{C}$ players 1, 2 and 3 belong to the same coalition.

First of all, notice that, for any $i \geq 4$, $v_{i,j} = 0$ for any $j \in [n]$; it follows that it is possible to discard players $4,\ldots,n$ in the following discussion. The outcome in which players 1, 2 and 3 belong to three different coalitions is not Nash stable because, for instance, player 1 would increase her utility from 0 to $1+\alpha$ by joining the coalition of player 2. The outcome in which players 1 and 2 are in a coalition and player 3 in another one is not Nash stable because player 3 would increase her utility from α to $2\alpha-\epsilon$ by joining the coalition of players 1 and 2 (notice that a symmetric argument holds for the outcome in which players 2 and 3 are in a coalition and player 1 in another one). Finally, the outcome in which players 1 and 3 are in a coalition and player 2 in another one is not Nash stable because, for instance, player 1 would increase her utility from $-1-\epsilon$ to 0 by forming alone a new coalition. It follows that in any Nash equilibrium $\mathcal{C}$ (recall that by Theorem 1 a Nash equilibrium always exists) players 1, 2 and 3 belong to the same coalition.

It can be easily verified that $u_1(\mathcal{C}) = u_3(\mathcal{C}) = 2\alpha - \epsilon$ and $u_2(\mathcal{C}) = 2 - 2\alpha\epsilon$. Moreover, since $\overline{u}_3(\mathcal{C}) = -\epsilon$, it follows that for any $i \geq 4$, $u_i(\mathcal{C}) = -\alpha\epsilon$.



(i,j)	$v_{i,j}$
(1,2)	1
(1,3)	$-1-\epsilon$
(2,3)	1

Fig. 3. Graph G and valuations $v_{i,j}$.

Therefore,

$$\mathsf{SW}(\mathcal{C}) = 2(2\alpha - \epsilon) + 2 - 2\alpha\epsilon - (n-3)\alpha\epsilon < 0$$

for n going to infinite.

In order to complete the proof, it is sufficient to note that there exists a coalition structure (for instance the one in which players 1 and 2 belong to the same coalition and all other players are alone in different coalitions) with positive social welfare. $\square$

Now we focus our attention on the $\overline{\mathsf{SW}}$ social welfare function and we show that $\overline{\mathsf{PoS}}$ is unbounded for α tending to 1.

Theorem 7. *Given any $M > 0$, there exist a value of α, a graph G and a function v (admitting also negative valuations) such that $\overline{\mathsf{PoS}}(\mathcal{G}(G, v, \alpha)) > M$.*

6 Open Problems

Our work leads to many future research directions. It would be interesting to analyze if it is possible to decrease the price of anarchy by restricting to particular classes of graphs, as well as to study different stability notions such as strong Nash outcomes and core stable outcomes. Moreover, the problem in which valuations can be different from zero only between players i, j for which edge $(i, j) \in E$ (and not for all pair of players) is worth studying. Finally, it would be interesting to study the fractional version of hedonic games, in which the utility of a player is divided by the number of players in the coalition she belongs to.

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